MTH 132 Exam 1 Topics

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Rates of Change

You should be familiar with computing the average rate of change on an interval for a given function, and relate it to the slope of the secant line of the function. So if we're given a function like

$$f(x) = x^2$$

and an interval $[1, 1+h]$, the average rate of change of f on the given interval is, by definition

$$\frac{\Delta f}{\Delta x} = \frac{f(1+h) - f(1)}{(1+h) - 1}$$

In general, we can always compute the average rate of change using this formula. We can simplify this using some algebra:

$$\begin{aligned}\frac{f(1+h) - f(1)}{(1+h) - 1} &= \frac{(1+h)^2 - 1^2}{h} \\ &= \frac{1 + 2h + h^2 - 1}{h} \\ &= \frac{2h + h^2}{h} \\ &= 2 + h\end{aligned}$$

We can interpret this number in terms of the *secant line*, which connects the point $(1, 1)$ with $(1+h, f(1+h))$ - the line has slope exactly $2 + h$. In the limit, this becomes the *tangent line*, which has slope

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} 2 + h = 2$$

So we'd say that the slope of the function at the point $(1, 1)$ is 2.

Limit Calculations

You should be able to compute some basic limits using continuity, limit laws, and algebraic manipulations. Let's study an example, of

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 7x + 6}$$

This is a quotient of polynomials, so it's continuous on its domain - the problem is that 1 isn't in the domain of the given function, since we'd be left with $0/0$. But we can "remove" the problem by factoring; everywhere on the domain of the given function, we have

$$\frac{x^2 - 1}{x^2 - 7x + 6} = \frac{(x-1)(x+1)}{(x-6)(x-1)} = \frac{x+1}{x-6}$$

Remember that when taking limits, we don't care about the function value or behaviour *at the point* - so this manipulation is valid; do notice, though, that the functions $(x^2 - 1)/(x^2 - 7x + 6)$ and $(x+1)/(x-6)$ aren't the same, since they have different domains. What is true is that

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 7x + 6} = \lim_{x \to 1} \frac{x+1}{x-6} = \frac{1+1}{1-6} = \frac{2}{-5}$$

Being able to eliminate division by zero is the first step in trying to calculate many of these limits.

Another algebraic technique is to *multiply by the conjugate*: Suppose we have a limit like

$$\lim_{t\to 1} \frac{\sqrt{3+t}-2}{t-1}$$

Again, there's division by zero; since we can't factor or divide by anything obvious, let's multiply. The use of this is to introduce a difference of squares, which we can hopefully simplify: Switching the sign in the numerator, we have

$$\frac{\sqrt{3+t}-2}{t-1} = \frac{\sqrt{3+t}-2}{t-1} \cdot \frac{\sqrt{3+t}+2}{\sqrt{3+t}+2} = \frac{(3+t)-4}{(t-1)(\sqrt{3+t}+2)} = \frac{1}{\sqrt{3+t}+2}$$

for all t in the domain we're looking at. This limit can then be computed by direct evaluation, giving $1/(\sqrt{4}+2) = 1/4$.

Let's look at another limit:

$$\lim_{x\to 0} \frac{\sin(6x)}{2x}$$

Certainly we can't evaluate this at 0, since we'd be left with $0/0$ again. But we can use the fact that

$$\lim_{t\to 0} \frac{\sin t}{t} = 1$$

Our first step is matching the numerator and the denominator; we need a 6 in the denominator to match the 6 in the numerator. This is done by

$$\frac{\sin(6x)}{2x} = \frac{\sin(6x)}{6x} \cdot 3$$

Now we can take the limit, pulling the constant out:

$$\lim_{x\to 0} \frac{\sin(6x)}{2x} = 3 \lim_{x\to 0} \frac{\sin(6x)}{6x} = 3 \cdot 1 = 3$$

This is the usual technique for dealing with limits with quotients involving the sin function.

WARNING: It is tempting, but not valid to write $\sin(6x) = 6\sin x$. Although this manipulation leads to the correct answer, it's not true.

For a more complicated example, we can study things like

$$\lim_{x\to 0} \frac{\sin(\sin x)}{x}$$

Again, this is resolved by matching the numerator and denominator:

$$\lim_{x\to 0} \frac{\sin(\sin x)}{x} = \lim_{x\to 0} \frac{\sin(\sin x)}{\sin x} \frac{\sin x}{x} = \left(\lim_{x\to 0} \frac{\sin(\sin x)}{\sin x}\right) \cdot \left(\lim_{x\to 0} \frac{\sin x}{x}\right) = 1 \cdot 1 = 1$$

Many of these same techniques can be used for studying limits as $x \to \pm\infty$, or for dealing with one-sided limits. One final technique is to make a change of variables to transform limits at ∞ to limits at 0; suppose we want to compute

$$\lim_{x\to -\infty} x \sin\left(\frac{1}{x}\right)$$

It's not clear how to proceed: There's not a rational function here, nor can we factor. But let's transfom it, by noting that

$$x \to -\infty \iff \frac{1}{x} \to 0^-$$

This suggests setting $t = \frac{1}{x}$ (so that $x = \frac{1}{t}$) and instead considering

$$\lim_{x\to -\infty} x \sin\left(\frac{1}{x}\right) = \lim_{t\to 0^-} \frac{1}{t} \sin t$$

But this final limit is just 1.

Another useful technique (that applies to all sorts of limits) is the Sandwich (Squeeze) Theorem; as an example, suppose that we have some random function f satisfying

$$1 - \cos x \leq f(x) \leq x^2$$

for all x in an interval around 0, except maybe at 0. Then since

$$\lim_{x \to 0} 1 - \cos x = 0 = \lim_{x \to 0} x^2$$

it follows that f has a limit at 0, and

$$\lim_{x \to 0} f(x) = 0$$

So it doesn't matter how badly behave f is, but only that f can be bounded by two "nice" functions. This result can be used for limits at infinity as well.

Formal Definition of the Limit

You should know the formal definition of a limit in various contexts, and how to apply it. For example, given an $\epsilon > 0$, you should be able to find $\delta > 0$ satisfying the definition of a limit.

For example, consider the function $g(x) = \frac{1}{x}$ near $x = 1$, and $\epsilon = .5$. We want to find a $\delta > 0$ such that

$$0 < |x - 1| < \delta \implies |g(x) - 1| < .5$$

We can look at the endpoints, when we have error exactly equal to .5:

$$g(x) - 1 = .5 \implies g(x) = 1.5 \implies g(x) = \frac{2}{3}$$

and

$$-(g(x) - 1) = .5 \implies -g(x) + 1 = .5 \implies g(x) = .5 \implies x = 2$$

So whenever $\frac{2}{3} < x < 2$, we have that $|g(x) - 1| < .5$. Choosing the smallest difference from 1, we'd set $\delta = \frac{1}{3}$, since the interval $(1 - 1/3, 1 + 1/3$ is the largest interval symmetric about 1, and contained in $(2/3, 2)$.

Given a "nice" function like a line, you should be able to find the largest δ corresponding to ϵ in the formal definition. For example, suppose we want to prove that

$$\lim_{x \to 2} 2x = 4$$

We'd want to find δ such that

$$0 < |x - 2| < \delta \implies |2x - 4| < \epsilon$$

Rearranging this a bit, this is the same as

$$0 < |x - 2| < \delta \implies |x - 2| < \frac{\epsilon}{2}$$

Since statements always imply themselves, we'd like these to be the same statement - we can make them the same by setting $\delta = \epsilon/2$. Once we've finished this scratchwork, we can being the proof:

Let $\epsilon > 0$ be given, and $\delta = \epsilon/2$. Then if $0 < |x - 2| < \delta$, we have

$$|2x - 4| = 2|x - 4| < 2\delta = \epsilon$$

as desired.

Remember that $\epsilon - \delta$ problems will generally have two parts: Scratchwork to find the best choice of δ in terms of ϵ, followed by a verification that that choice of δ *actually works*.

Continuity If a function is continuous at c, we can compute limits easily, since we have

$$\lim_{x \to c} f(x) = f(c)$$

You should know how to find where a function is continuous, and explain why; most of the functions we have dealt with, including

- Polynomials
- Rational functions
- Trig functions (sine, cosine, tangent, ...)
- Absolute value

are continuous on their domains. Remember to check for division by zero, though - a quotient of continuous functions is not necessarily continuous when the denominator goes to zero.

Now one of the best tools that we have for dealing with continuous functions is the *Intermediate Value Theorem*; it says that between a and b, a continuous function f must take on every value between $f(a)$ and $f(b)$; so for example, if we have a continuous function f satisfying $f(0) = 0$ and $f(1) = 1$, then there's a value c with $f(c) = .92$. We can use this to show that equations have solutions; for example, suppose we wanted to solve

$$x^{80} + x^{79} = 1 + \cos x$$

This is *very* difficult to solve analytically (if not impossible). But if we define

$$f(x) = 1 + \cos x - x^{80} - x^{79}$$

then we have that

$$f(0) = 2 > 0 \qquad f(1) = \cos 1 - 1 < 0$$

So there's a value α between 0 and 1 for which $f(\alpha) = 0$; rearranging this, α is a solution to the original equation.

Limits Involving Infinity Many of the techniques discussed above can be extended to limits as $x \to \pm\infty$, or when $\lim_{x\to a} f(x) = \pm\infty$. In particular, you should know how to find all kinds of asymptotes (vertical, horizontal, and oblique). For example, since

$$\lim_{x\to\infty} \frac{x^4 + 3x^2}{x^4 - 22x + 60} = \lim_{x\to\infty} \frac{1 + \frac{3}{x^2}}{1 - \frac{22}{x^3} + \frac{60}{x^4}} = 1$$

the function $f(x) = (x^4 + 3x^2)/(x^4 - 22x + 60)$ has a horizontal asymptote at $y = 1$. In general, we have some rules for rational functions about what sort of asymptotes are possible:

- If the degree of the numerator is $\leq$ the degree of the denominator, there is a horizontal asymptote. The asymptote is zero if we have $<$, and is the quotient of the leading coefficients if we have $=$.
- If the degree of the numerator is one more than the degree of the denominator, there is an oblique asymptote.
- Whenever the denominator of a rational function is zero, there may be a vertical asymptote (and this is necessary); it's not guaranteed, though. The function

  $$\frac{x-1}{x^2-1}$$

  has a vertical asymptote at $x = -1$, but only a removable discontinuity at $x = 1$. On the other hand,

  $$\frac{x}{x^2-1}$$

  has vertical asymptotes at both ± 1.

This isn't a complete list of topics for the exam, but it's a start. Make sure you can draw the pictures for various concepts, e.g. limits and the intermediate value theorem - they can help guide your intuitition in problem solving. I'd also suggest doing as many WeBWorK problems as possible, as well as problems from the Chapter 2 Review in the textbook. Any material from Sections 2.1-2.6 is fair for the exam.